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A SHORT PROOF OF ALIENATION OF ADDITIVITY FROM QUADRATICITY

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ABSTRACT. Without the use of pexiderized versions of abstract polynomials theory, we show that on 2-divisible groups the functional equation

$$f(x+y) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + 2g(y)$$

forces the unknown functions f and g to be additive and quadratic, respectively, modulo a constant.

Motivated by the observation that the equation

$$f(x+y) + f(x^2) = f(x) + f(y) + f(x)^2$$

implies both the additivity and multiplicativity of f , we deal also with the alienation phenomenon of equations in a single and several variables.

1. Alienation phenomenon

Let $(G, +)$ and $(H, +)$ be groups (not necessarily commutative). A map $A : G \rightarrow H$ is called *additive* provided that it satisfies Cauchy's functional equation

$$A(x+y) = A(x) + A(y) \quad \text{for all } x, y \in G,$$

whereas a map $Q : G \rightarrow H$ is termed *quadratic* whenever

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \quad \text{for all } x, y \in G$$

(see, e.g., J. A c z é l & J. D h o m b r e s [1]).

Summing up these two fundamental functional equations side by side, we get $A(x+y) + Q(x+y) + Q(x-y) = A(x) + A(y) + 2Q(x) + 2Q(y)$ for all $x, y \in G$.

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It seems hardly likely that given two maps $f, g : G \rightarrow H$ the corresponding equation

$$f(x+y) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + 2g(y), \quad x, y \in G,$$

brings us back to the additivity of f and hence the quadraticity of g (alienation phenomenon). Nevertheless, bearing in mind the results obtained by J. Dhombrès [2] (where the alienation idea comes from) and by the present author in [3] and [4] (see also R. Ger & L. Reich [5]), such a conjecture becomes more reasonable. Two years ago, jointly with M. Sablik, the author published the survey article [6] on the alienation phenomenon in the theory of functional equations. The reference list therein contains several dozen of items concerning alienation.

In contrast to the papers just quoted, we have decided to discuss a Pexider version of the problem from the very beginning, because in the case, where $f = g$, i.e., in the case of equation

$$2f(x+y) + f(x-y) = 3f(x) + 3f(y) \quad \text{for all } x, y \in G,$$

one may easily check that it admits constant solutions only (at least whenever the domain group is 2-divisible).

Therefore, our chief concern is to examine whether or not the functional equation

$$f(x+y) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + 2g(y), \quad x, y \in G,$$

is equivalent to the system of two fundamental functional equations

$$\begin{cases} f(x+y) = f(x) + f(y), \\ g(x+y) + g(x-y) = 2g(x) + 2g(y) \end{cases} \quad \text{for every } x, y \in S.$$

Nine years ago, on August 30, 2010, during the 24th International Summer Conference on Real Functions Theory, the author presented a talk entitled

“Additivity and exponentiality are alien to each other”.

As a matter of fact the present paper might have an analogous title

“Additivity and quadraticity are alien to each other”

as well. The point is that the latter statement may be derived from what is known in the theory of linear equations of the form

$$\sum_{i=1}^n f_i(a_i x + b_i y) = 0, \quad (*)$$

where the unknown functions f_i are mappings between two given groups and a_i, b_i are integers for $i \in \{1, \dots, n\}$.

In fact, the equation we are dealing with has the form

$$f(x+y) - f(x) - f(y) + g(x+y) + g(x-y) - 2g(x) - 2g(y) = 0 \in G,$$

and we have here

$$n = 7, \quad f_1 = f, \quad f_2 = f_3 = -f, \quad f_4 = f_5 = g, \quad f_6 = f_7 = -2g$$

and

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 0, \quad a_4 = 1, \quad a_5 = 1, \quad a_6 = 1, \quad a_7 = 0,$$

$$b_1 = 1, \quad b_2 = 0, \quad b_3 = 1, \quad b_4 = 1, \quad b_5 = -1, \quad b_6 = 0, \quad b_7 = 1.$$

However, that derivation (from papers of A. Lisak and M. Sablik [9], M. Sablik [10] and [11] or L. Székelyhidi [12]) requires some work to be done. Moreover, all of these authors assume the commutativity of the domain spoken of, whereas we do not.

2. Main result

Our main result establishes the alienation phenomenon of additivity and quadraticity up to a constant.

THEOREM 2.1. *Let $(G, +)$ be a 2-divisible group (not necessarily commutative) and let $(H, +)$ be an Abelian group. Then, functions $f, g : G \rightarrow H$ satisfy the functional equation*

$$f(x+y) + g(x+y) + g(x-y) = f(x) + f(y) + 2g(x) + 2g(y) \quad \text{for all } x, y \in G \quad (\text{E})$$

if and only if there exist: an additive function $A : G \rightarrow H$, a quadratic function $Q : G \rightarrow H$, and a constant $c \in H$ such that

$$f(x) = A(x) - 2c \quad \text{and} \quad g(x) = Q(x) + c \quad \text{for all } x \in G.$$

In other words, additivity and quadraticity are alien to each other modulo a constant.

Proof. A straightforward verification shows that for any additive function $A : G \rightarrow H$, any quadratic function $Q : G \rightarrow H$ and any constant $c \in H$ the maps $f := A - 2c$ and $g := Q + c$ yield a solution of equation (E).

Conversely, assuming the validity of (E), put $F := f + g$ to get

$$F(x+y) + g(x-y) = F(x) + F(y) + g(x) + g(y) \quad \text{for all } x, y \in G.$$

Let

$$\Phi(x, y) := F(x+y) - F(x) - F(y) = g(x) + g(y) - g(x-y), \quad x, y \in G.$$

Then Φ , as a Cauchy difference, satisfies the cocycle equation

$$\Phi(x+y, z) + \Phi(x, y) = \Phi(x, y+z) + \Phi(y, z) \quad \text{for all } x, y, z \in G.$$

Consequently,

$$\begin{aligned} g(x+y) + g(z) - g(x+y-z) + g(x) + g(y) - g(x-y) \\ = g(x) + g(y+z) - g(x-z-y) + g(y) + g(z) - g(y-z) \end{aligned}$$

for all $x, y, z \in G$, i.e.,

$$g(x+y) - g(x+y-z) - g(x-y) = g(y+z) - g(x-z-y) - g(y-z). \quad (1)$$

Replacing here z by $-z$, we infer that

$$g(x+y) - g(x+y+z) - g(x-y) = g(y-z) - g(x+z-y) - g(y+z). \quad (2)$$

Now, subtract equalities (1) and (2) side by side to get

$$g(x+y+z) - g(x+y-z) = 2g(y+z) - 2g(y-z) + g(x+z-y) - g(x-z-y)$$

which, after setting $z = y$, leads to

$$g(x+2y) - g(x) = 2g(2y) - 2g(0) + g(x) - g(x-2y),$$

whence

$$g(x+2y) + g(x-2y) = 2g(x) + 2g(2y) - 2g(0), \quad x, y \in G.$$

Due to the 2-divisibility of $(G, +)$ on putting $c := g(0)$, we deduce that

$$g(x+y) + g(x-y) = 2g(x) + 2g(y) - 2c, \quad x, y \in G,$$

which states that the function $Q := g - c$ is *quadratic*.

Going back to the original equation (E), we obtain

$$f(x+y) + Q(x+y) + c + Q(x-y) + c = f(x) + f(y) + 2Q(x) + 2c + 2Q(y) + 2c$$

for all $x, y \in G$ whence

$$f(x+y) = f(x) + f(y) + 2c, \quad x, y \in G,$$

which states that the function $A := f + 2c$ is *additive*.

Thus,

$$f(x) = A(x) - 2c, \quad g(x) = Q(x) + c \quad \text{for all } x \in G,$$

as it is claimed. □

3. Some remarks on alienation of equations in several variables and iterative ones

Motivated by the observation (folklore) that the equation

$$f(x+y) + f(x^2) = f(x) + f(y) + f(x)^2 \quad (3)$$

implies both the additivity and multiplicativity of a function f mapping a ring into a ring with no elements of order 2, we may ask, for instance, whether or not the fundamental equations in the theory of functional equations in several

variables and that of iterative functional equations (the Cauchy equation and the Schröder equation, respectively) are alien. That is we ask whether the equation

$$f(x+y) + f(\varphi(x)) = f(x) + f(y) + sf(x) \quad (4)$$

forces f to be an additive solution to the Schröder equation $f(\varphi(x)) = sf(x)$ (see, e.g., M. K u c z m a [7] or M. K u c z m a, B. C h o c z e w s k i and R. G e r [8]). Equation (4) results from the standard alienation procedure starting in this case from adding the additivity equation and the Schröder equation side by side. Now, both of the equations (3) and (4) have the form

$$f(x+y) - f(x) - f(y) = F(x),$$

where F is a function in a single variable. By interchanging the roles of x and y here, we infer (at least in the case of commutative domains) that F has to be constant, say c . In our case, we have

$$F(x) = sf(x) - f(\varphi(x)) \equiv c.$$

This forces f to be affine, i.e., $f(x) = A(x) - c$, where A stands for an additive map. Consequently, a modified Schröder equation we arrive at has the form

$$A(\varphi(x)) = sA(x) - sc$$

with the unknown additive function A . Clearly, the solutions depend on the domains and ranges considered. On the real line, for rational s , in the case where the given function φ is continuous and nonlinear, the situation turns out to be trivial. In more abstract structures the latter equation presents an interest of its own.

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